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Existence and Uniqueness of Solutions to a Class of Nonlinear-Operator-Differential Equations Arising in Automatic Spaceship Navigation

Victor M. Bogdan

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Existence and Uniqueness of Solutions to a Class of Nonlinear-Operator-Differential Equations Arising in Automatic Spaceship Navigation

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1.0 INTRODUCTION

Consider the differential equation

$$r''(t) = G(r(t)) + u(t) \tag{1-1}$$

describing the motion of a spaceship, where r is the position vector, $G(r)$ is the gravitational acceleration, and u denotes the thrust acceleration.

Let us introduce a six-dimensional vector $y = (r,v)$ where $v = r'$. Then equation (1-1) can be written in the form

$$y' = f(t,y,u) \tag{1-2}$$

where

$$f(t,y,u) = (v,\ G(r) + u) \tag{1-3}$$

Since the velocity of any motion does not exceed the speed of light and the gravitational acceleration for motions (in a central gravitational field) which are bounded away from the center can be considered as bounded, the resulting function f can be assumed to be bounded, at least for physical motions, and the gravitational acceleration can be written in the form

$$G(r) = -k(r)r \tag{1-4}$$

where

$$k(r) = \mu/r^3 \quad \text{if} \quad r \geq r_o \tag{1-5}$$

and

$$k(r) = \mu/r_o^3 \quad \text{if} \quad r \leq r_o \tag{1-6}$$

This modification of the gravitational acceleration will not change the motion provided that the motion is outside the sphere with radius r_o with its center at the origin.

It is easy to prove that after such a modification the function G is Lipschitzian and its smallest Lipschitz constant (as discussed by V. M. Bogdan and V. R. Bond in a paper entitled "A priori Global Estimates of Deviation of Solutions of Differential Equations Due to Perturbation With Applicatons to Analytical and Celestial Mechanics") is given by the formula

$$\|G\| = (2\mu)/r_o^3 \tag{1-7}$$

Notice that in the above argument it is not essential that the central gravitational field has its center at the origin. The center of the field could be in motion and the above regularization procedure would still apply provided that the distance between the center and the spaceship is always larger than r_o. Notice also that if the gravitation acceleration function is the sum of several central Newtonian fields then the above procedure applied to each of them will regularize the function G to make it Lipschitzian. For details see the paper referenced in the previous paragraph.

Let us assume that the automatic control is given by the formula

$$u(t) = g(t, y(t), t_1, y(t_1)) \tag{1-8}$$

where t_1 denotes the time that the telemetric reading of the position-velocity vector $y(t_1)$ is taken. We assume that this information is transmitted to the control center, which is located at the origin of an inertial coordinate system. At the control center, this telemetric information is used to produce a computer program which is then transmitted to the spaceship. The program represents a function such that, when the program reaches the spaceship, new telemetric readings of the time and the vector $y(t)$ are taken, and the program is executed, one obtains as a result the value of the control vector given by formula (1-8). We assume that this process is performed continuously, as by an analog computer. We shall be interested in such motions when the transmission time is not negligible, as when the spaceship is far away from the control center.

Our first objective will be to find the time t_1 and the position-velocity vector $y(t_1)$ as an operator of the trajectory function y.

The mathematical terminology and notions to be used in this paper can be found in the books Optimal Control by Athians and Falb (ref. 2) and Foundations of Modern Analysis by Dieudonne (ref. 4). For extensive literature on delay differential equations and related topics, see the survey by Corduneanu and Lakshmikantham (ref. 3). For regularization of the gravitational acceleration used in this paper and explicit formulas for computing the smallest Lipschitzian constant for a vector function, see the aforementioned paper by Bogdan and Bond. Valuable information on delay differential equations is contained in Driver (ref. 5) and Hale (ref. 6). Papers by Anderson and Bogdanowicz (ref. 1) and Lakshmikantham, Leela, and Moauro (ref. 7) are closely related to the results presented in this paper.

2.0 TIME DELAY OPERATOR

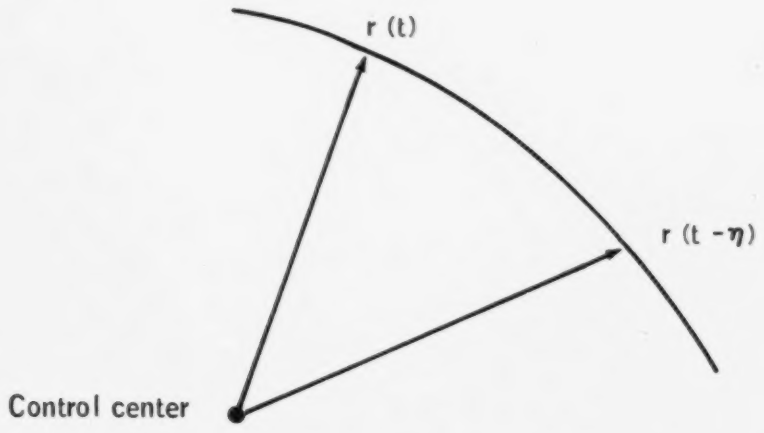
Assume that a trajectory is defined by

$$y(t) = (r(t), v(t)) \tag{2-1}$$

for all values of time. Let us fix time t and denote by $\eta = t - t_1$ the delay caused by the transmission of the telemetric data to the control center and of the program back to the spaceship. This delay can be represented by the formula

$$\eta = p + |r(t)|/c + |r(t - \eta)|/c \tag{2-2}$$

where p denotes the processing time required by the computers and c the velocity of light. (See the following sketch.)



Let us assume that for some number

$$0 < q < 1 \tag{2-3}$$

every admissible physical motion is such that

$$|v(t)| \leq cq \tag{2-4}$$

for any value of t.

Under this assumption, we have the following estimate

$$\begin{aligned} \eta &\leq p + |r(t)|/c + |r(t) - \int_{t-\eta}^{t} v(s)\, ds|/c \\ &\leq p + 2|r(t)|/c + \eta q \end{aligned} \tag{2-5}$$

which yields the inequality

$$\eta \leq (1 - q)^{-1}(p + 2|r(t)|/c) = a(t) \tag{2-6}$$

Consider a function s defined by

$$s(\eta) = p + |r(t)|/c + |r(t - \eta)|/c \tag{2-7}$$

for all $\eta \geq 0$. Observe that the function s maps the closed interval

$$I = \langle 0, a(t) \rangle \tag{2-8}$$

into I and it represents a contraction of this interval. Indeed, we have the inequality

$$|s(\eta) - s(\rho)| \leq (1/c) \int_{t-\rho}^{t-\eta} |v(x)|\, dx \leq q|\eta - \rho| \tag{2-9}$$

which proves that the function s is a contraction. Thus, there exists a unique number η satisfying equality (2-2) for every trajectory function y satisfying condition (2-4).

This means that the time t_1 can be treated as an operator of the trajectory function y and is given by the formula

$$t_1 = T_1(y)(t) = t - \eta \tag{2-10}$$

3.0 OPERATORS ASSOCIATED WITH AUTOMATIC CONTROL OF SPACESHIPS

In this section, we will investigate the operator T_1 and the operator defined by the formula

$$T_2(y)(t) = y(T_1(y)(t)) \tag{3-1}$$

for t belonging to the interval I and y belonging to a suitable space of continuous functions. Using these operators, we can convert the original spaceship control problem to the following form:

$$\left.\begin{aligned} y'(t) &= f(t, y(t), u(t)) \qquad (3\text{-}2)\\ u(t) &= g(t, y(t), T_1(y)(t), T_2(y)(t)) \qquad (3\text{-}3)\end{aligned}\right\} \text{for all } t \geq 0$$

$$y(t) = x(t) \qquad \text{for all } t \leq 0 \qquad (3\text{-}4)$$

where the function x is assumed to be known.

Notice that a physical motion must satisfy the following Lipschitzian constraint

$$|v(t) - v(t_1)| \leq a|t - t_1| \tag{3-5}$$

since the acceleration of any spaceship cannot exceed a certain limit at which destructive stresses will be produced. Also, the fact that the velocity of the spaceship cannot exceed the velocity of light yields another constraint

$$|r(t) - r(t_1)| \leq c|t - t_1| \tag{3-6}$$

for any t and t_1.

In the sequel, we shall denote by I an interval of the form

$$\langle 0,b\rangle \quad \text{or} \quad \langle 0,\infty) \tag{3-7}$$

Let Y be a Banach space and let x be a Lipschitzian function with Lipschitz constant w from $(-\infty,0\rangle$ into Y. We shall denote by

$$L_{xw}(Y) \tag{3-8}$$

the space of all functions y from the interval $(-\infty,\infty)$ into the Banach space Y such that

$$|y(t) - y(t_1)| \leq w|t - t_1| \tag{3-9}$$

for all $t,t_1 \in I$ and

$$y(t) = x(t) \tag{3-10}$$

for $t \leq 0$.

Notice that this set of functions restricted to I can be considered as a subset of the space $C(Y)$ of all continuous functions y from the interval I into the Banach space Y. In the space $C(Y)$, we shall introduce a family of norms given by the formula

$$|y|_k = \sup \{e^{-kt}|y(t)| \,:\, t \in I\} < \infty \tag{3-11}$$

for any fixed real number k. Observe that from this definition follows the inequality

$$|y(t)| \leq |y|_k e^{kt} \tag{3-12}$$

for all $t \in I$.

Notice that when the interval I is compact all the norms (3-11) are equivalent. When the interval I is unbounded, each value of k yields a different subset $C_k(Y)$ of the space $C(Y)$ of all continuous functions.

Let Z_1, Z_2 be two Banach spaces. Let us assume that D is a subset of the space Z_1. We shall denote by

$$L(D, Z_2) \tag{3-13}$$

the space of all operators U from the set D into the Banach space Z_2 satisfying the Lipschitz condition; i.e., there exists a number L such that

$$|U(z) - U(z_1)| \leq L|z - z_1| \tag{3-14}$$

for all z and z_1 in the set D. We shall denote by $\|U\|$ the greatest lower bound of all the numbers L satisfying inequality (3-14). The value $\|U\|$ considered as a function of the operator U forms a seminorm on the space $L(D, Z_2)$.

Lemma. Let Z_1, Z_2, Z_3 be some Banach spaces and let D_i denote a subset of the space Z_i for $i = 1, 2$. If $U_i \in L(D_i, Z_{i+1})$ for $i = 1, 2$, then the composition operator $U = U_2U_1$ belongs to the space $L(D_1, Z_3)$ and the following inequality

$$\|U_2U_1\| \leq \|U_2\| \|U_1\| \tag{3-15}$$

holds. The proof of the lemma is obvious.

4.0 OPERATORS OF THE EXPONENTIAL TYPE

Assume that given are two Banach spaces Y_i $(i=1,2)$. For the sake of brevity, let us denote by $C_i = C(Y_i)$ the Banach space of all continuous functions from the interval I into the space Y_i. Let D_i denote a subset of the space C_i for $i = 1, 2$.

4.1 Definition

An operator U will be said to be of the exponential type if and only if it belongs to some space $L(D_1,C_2)$ of Lipschitzian operators and there exist a number b and a positive constant k_o such that $\|U\|_k \leq b$ for all $k > k_o$ where $\| \|_k$ denotes the seminorm induced on the space $L(D_1,C_2)$ by the norm $| |_k$ defined on the spaces C_i $(i=1,2)$ of continuous functions.

In the sequel, we shall assume that we are dealing with motions y from the interval $(-\infty,\infty)$ into the space $Y = R^6$ such that

$$y(t) = (r(t),\ v(t)) \tag{4-1}$$

and

$$y(t) = x(t) \tag{4-2}$$

for times $t \leq 0$.

4.2 Theorem

The operator T_1 considered from the set $L_{xw}(Y)$ into $C(R)$ is of the exponential type for any continuous function x from $(-\infty,0>$ into Y having Lipschitz constant $w < c$. Moreover,

$$\|T_1\|_k \leq 2/(c - w) \tag{4-3}$$

for all $k \geq 0$.

Proof. Take a fixed argument $t \in I$ and two functions

$$y_i \in L_{xw}(Y) \tag{4-4}$$

for $i = 1, 2$, and let

$$y_i = (r_i,\ v_i) \tag{4-5}$$

be the decomposition into the radius vector component and the velocity vector. Let us introduce the notation

$$T_1(y_i)(t) = t - \eta_i \tag{4-6}$$

for a fixed value of t. Recall that the numbers η_i satisfy the equations

$$\eta_i = (|r_i(t)| + |r_i(t - \eta_i)|)/c + p \tag{4-7}$$

See equation (2-2).

Relations (4-6) and (4-7) yield the equality

$$|T_1(y_1)(t) - T_1(y_2)(t)| = |\eta_1 - \eta_2| \tag{4-8}$$

and thus we get the inequality

$$c|\eta_1 - \eta_2| \leq |r_1(t) - r_2(t)| + |r_1(t - \eta_1) - r_2(t - \eta_2)| \tag{4-9}$$

Let us estimate the second term on the right-hand side of inequality (4-9). We have from the triangle inequality

$$\begin{aligned} |r_1(t - \eta_1) - r_2(t - \eta_2)| \leq\ & |r_1(t - \eta_1) - r_1(t - \eta_2)| \\ & + |r_1(t - \eta_2) - r_2(t - \eta_2)| \end{aligned} \tag{4-10}$$

From the Lipschitz condition, the first term on the right-hand side of inequality (4-10) can be estimated by

$$|r_1(t - \eta_1) - r_1(t - \eta_2)| \leq w|\eta_1 - \eta_2| \tag{4-11}$$

Notice that the second term on the right-hand side of inequality (4-10) is equal to zero if

$$t - \eta_2 \leq 0 \tag{4-12}$$

and it can be estimated in any case by

$$\begin{aligned} |r_1(t - \eta_2) - r_2(t - \eta_2)| &\leq |r_1 - r_2|_k e^{k(t-\eta_2)} \\ &\leq |y_1 - y_2|_k e^{kt} \end{aligned} \tag{4-13}$$

as follows from the definition of the norm $|\ |_k$ (see formula (3-12)).

Since we also have the estimate

$$|r_1(t) - r_2(t)| \leq |y_1 - y_2|_k e^{kt} \tag{4-14}$$

for all $t \in I$, we get from estimates (4-10), (4-11), and (4-13) and inequality (4-9) the following estimate

$$(c - w)|\eta_1 - \eta_2| \leq 2|y_1 - y_2|_k e^{kt} \tag{4-15}$$

for all t in the interval I. Multiplying both sides of inequality (4-15) by e^{-kt} and dividing by $c - w$, we get

$$e^{-kt}|\eta_1 - \eta_2| \leq b|y_1 - y_2|_k \tag{4-16}$$

for all t in the interval I, where b denotes the constant

$$b = 2/(c - w) \tag{4-17}$$

Taking the supremum over all $t \in I$ on the left-hand side of inequality (4-16), we get the inequality

$$|T_1(y_1) - T_1(y_2)|_k \leq b|y_1 - y_2|_k \tag{4-18}$$

for all y_1, y_2 from the set $L_{xw}(Y)$ and for $k \geq 0$. Hence, we have the estimate

$$\|T_1\|_k \leq b \tag{4-19}$$

for all $k \geq 0$. This completes the proof.

4.3 Theorem

The operator T_2 defined by formula (3-1) is of the exponential type from the set $L_{xw}(Y)$ into $C(Y)$ for every fixed function x with Lipschitz constant $w < c$. Moreover,

$$\|T_1\|_k \leq (c + w)/(c - w) \tag{4-20}$$

for all $k \geq 0$.

Proof. Take any fixed $t \in I$, $y_1, y_2 \in L_{xw}(Y)$, and notice the inequality

$$\begin{aligned} |T_2(y_1)(t) - T_2(y_2)(t)| &= |y_1(T_1(y_1)(t)) - y_2(T_1(y_2)(t))| \\ &\leq |y_1(T_1(y_1)(t)) - y_1(T_1(y_2)(t))| \\ &\quad + |y_1(T_1(y_2)(t)) - y_2(T_1(y_2)(t))| \end{aligned} \tag{4-21}$$

The first term on the right-hand side of inequality (4-21) can be estimated from the Lipschitz condition as follows

$$\begin{aligned} |y_1(T_1(y_1)(t)) - y_1(T_1(y_2)(t))| &\leq w|T_1(y_1)(t) - T_1(y_2)(t)| \\ &\leq w\|T_1\|_k|y_1 - y_2|_k e^{kt} \\ &\leq (2w/(c - w))|y_1 - y_2|_k e^{kt} \end{aligned} \tag{4-22}$$

The second term on the right-hand side of inequality (4-21)

$$|y_1(T_1(y_2)(t)) - y_2(T_1(y_2)(t))|$$

is equal to zero if the value $T_1(y_2)(t)$ is negative and it can be estimated in any case by the quantity

$$|y_1 - y_2|_k e^{kt} \tag{4-23}$$

as follows from the definition of the norm $|\ |_k$ (see formula (3-12)).

From inequality (4-21), we get the estimate

$$|T_2(y_1) - T_2(y_2)|_k \leq b|y_1 - y_2|_k \tag{4-24}$$

where

$$b = (c + w)/(c - w) \tag{4-25}$$

Hence,

$$\|T_2\|_k \leq b \tag{4-26}$$

for all $k \geq 0$. This completes the proof.

5.0 LIPSCHITZIAN FUNCTIONS INDUCE OPERATORS OF THE EXPONENTIAL TYPE

Let us assume that Y, Z denote two Banach spaces. Let f be a continuous function from the product $I \times Y$ into the space Z such that

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2| \tag{5-1}$$

for all $t \in I$ and all $y_1, y_2 \in Y$. Moreover, let us assume that the function satisfies the inequality

$$|f(t, y)| \leq L \tag{5-2}$$

for all $t \in I$ and all $y \in Y$. The infimum of all such constants L appearing in inequalities (5-1) and (5-2) will be denoted by $\|f\|$. Denote by

$$Lip(I \times Y, Z) \tag{5-3}$$

the set of all such functions. It is easy to prove that this set of functions with the norm $\|f\|$ as defined above forms a Banach space.

For a positive number w, we define the set

$$B_w(Z) = \{z \in C(Z) : |z(t)| \leq w\} \tag{5-4}$$

Lemma. If $f \in \mathrm{Lip}(I\times Y,Z)$, then the operator F, defined by $z = F(y)$ if and only if

$$z(t) = f(t,y(t)) \tag{5-5}$$

for all $t \in I$ and $y \in C(Y)$, is of the exponential type from the space $C(Y)$ into the set $B_w(Z)$, where $w = \|f\|$. Moreover, the estimate

$$\|F\|_k \leq \|f\| \tag{5-6}$$

holds for all $k \geq 0$.

Proof of the lemma. Notice that the operator F maps the set $C(Y)$ into $B_w(Z)$. To prove that it is of the exponential type, take any two functions

$$y_1, y_2 \in C(Y) \tag{5-7}$$

From the inequality

$$\begin{aligned} |f(t,y_1(t)) - f(t,y_2(t))| &\leq \|f\| \, |y_1(t) - y_2(t)| \\ &\leq \|f\| \, |y_1 - y_2|_k e^{kt} \end{aligned} \tag{5-8}$$

valid for all t belonging to the interval I, we get the inequality

$$|F(y_1) - F(y_2)|_k \leq \|f\| \, |y_1 - y_2|_k \tag{5-9}$$

for all functions $y_1, y_2 \in C(Y)$ and all $k \geq 0$. This completes the proof.

6.0 PROPERTIES OF A VOLTERRA INTEGRAL OPERATOR

We shall define the operator J by the condition

$$z = J(y) \tag{6-1}$$

if and only if

$$z(t) = x(0) + \int_0^t y(s)\, ds \tag{6-2}$$

for every $t \in I$ and

$$z(t) = x(t) \tag{6-3}$$

for every $t \leq 0$ and every $y \in L_{xw}(Y)$.

Theorem. The Volterra integral operator J is of the exponential type from every set $B_w(Y)$ into the set $L_{xw}(Y)$ for any fixed function x with Lipschitz constant w. Moreover,

$$\|J\|_k \leq 1/k \tag{6-4}$$

holds for all $k > 0$.

Proof. The operator J is well defined from the set $B_w(Y)$ into the set $L_{xw}(Y)$. To prove that this operator is of the exponential type, take any two functions $y_1, y_2 \in B_w(Y)$. Notice the inequality

$$\begin{aligned} |J(y_1)(t) - J(y_2)(t)| &\leq \int_0^t |y_1(s) - y_2(s)|\, ds \\ &\leq \int_0^t |y_1 - y_2|_k e^{ks}\, ds \\ &\leq (1/k)e^{kt}\, |y_1 - y_2|_k \end{aligned} \tag{6-5}$$

Hence, we get the inequality

$$|J(y_1) - J(y_2)|_k \leq (1/k)\, |y_1 - y_2|_k \tag{6-6}$$

for any two functions y_1, y_2 from the set $B_w(Y)$. This completes the proof.

7.0 DYNAMICAL SYSTEMS WITH AUTOMATIC CONTROLS OF THE EXPONENTIAL TYPE

Consider a dynamical system with the state equation

$$y'(t) = f(t,y(t),u(t)) \tag{7-1}$$

where the control satisfies the equation

$$u(t) = U(y)(t) \tag{7-2}$$

for all $t \in I$. The following theorem establishes the existence and uniqueness of solutions of the system of equations (7-1) and (7-2) under the assumption that the function f is Lipschitzian and the operator U is of the exponential type.

Theorem. If $f \in Lip(I\times Y\times Z,Y)$ and the operator U is of the exponential type from the set $L_{xw}(Y)$ into $C(Z)$ for every function x from $(-\infty,0>$ into Y with Lipschitz constant $w = \|f\|$, then for every such function x there exists a unique solution to the automatic control problem expressed by equations (7-1) and (7-2) such that $y(t) = x(t)$ for $t \leq 0$.

Proof. First, notice that every solution of the system of equations (7-1) and (7-2), if it exists, has to belong to the set $L_{xw}(Y)$, where $w = \|f\|$.

To prove the existence of the solution, notice that the system of equations with the initial condition $y(t) = x(t)$ for all $t \leq 0$ is equivalent to the following integral equation

$$y(t) = x(0) + \int_0^t f(s,\ y(s),\ U(y)(s))\ ds \tag{7-3}$$

to be satisfied for all $t \in I$. To analyze the integral equation, let us introduce the operator X from the space $L_{xw}(Y)$ into the space $C(Y\times Z)$ defined by means of the formula

$$X(y) = (y,\ U(y)) \tag{7-4}$$

for all $y \in L_{xw}(Y)$. Notice that the operator X is of the exponential type on every set $L_{xw}(Y)$.

Using the Volterra operator J and the operator F induced by the function f over the space $C(Y\times Z)$, we can write integral equation (7-3) in the equivalent form

$$y = JFX(y) \tag{7-5}$$

Notice that the operator JFX maps every set $L_{xw}(Y)$ into itself, where $w = \|f\|$, and that it is of the exponential type. Moreover, we have the estimate

$$\|JFX\|_k \leq (1/k)\ \|f\|\ b \tag{7-6}$$

for all $k > 0$, where b denotes a constant such that

$$\|X\|_k \leq b \tag{7-7}$$

for all $k > k_o$. Thus, if we select a value of k such that

$$k > \max\{\ b\|f\|,\ k_o\ \} \tag{7-8}$$

the operator JFX will become a contraction map as follows from formula (7-6). Since the set $L_{xw}(Y)$ can be considered as closed in the Banach space $C_k(Y)$, it represents a complete metric space. Hence, by the Banach contraction mapping theorem there exists a unique fixed point $y \in M_{xw}(Y)$ for the operator; i.e.,

$$y = JFX(y) \tag{7-5}$$

This completes the proof of the theorem.

8.0 APPLICATION TO AUTOMATIC CONTROL OF SPACESHIPS

Consider the original problem. Let us assume that we are given the motion of a spaceship

$$y'(t) = f(t,y(t),u(t)) = (v(t),\ G(r(t)) + u(t)) \tag{8-1}$$

with controls given by

$$u(t) = g(t, y(t), T_1(y)(t), T_2(y)(t)) \tag{8-2}$$

We assume that $Y = R^6$ and $Z = R^3$. The operators T_1, T_2 are defined as in the previous sections. We assume that the motion of the spaceship $y(t) = x(t)$ is known for $t \leq 0$.

Theorem. If $f \in Lip(I \times Y \times Z, Y)$ and $g \in Lip(I \times Y \times R \times Y, Z)$ and the operators T_1 and T_2 are defined as before, then for every function x from $(-\infty, 0>$ into Y with Lipschitz constant $w = \|f\| < c$ there exists a unique solution to the automatic control problem given by equations (8-1) and (8-2) such that

$$y(t) = x(t) \tag{8-3}$$

for $t \leq 0$.

Proof. The operator U, defined by the condition

$$u = U(y) \tag{8-4}$$

if and only if

$$u(t) = g(t, y(t), T_1(y)(t), T_2(y)(t)) \tag{8-5}$$

for all $t \in I$, maps the set $L_{xw}(Y)$ into the space $C(Y)$ of continuous functions and is of the exponential type. To prove this, take any two functions $y_1, y_2 \in L_{xw}(Y)$. We assume that $u_i = U(y_i)$ for $i = 1, 2$. Then we have the following estimates

$$\begin{aligned} |u_1(t) - u_2(t)| &\leq \|g\| \left(|y_1(t) - y_2(t)| + |T_1(y_1)(t) - T_1(y_2)(t)|\right. \\ &\qquad \left. + |T_2(y_1)(t) - T_2(y_2)(t)|\right) \\ &\leq b|y_1 - y_2|_k e^{kt} \end{aligned} \tag{8-6}$$

for all $t \in I$ and $y_1, y_2 \in L_{xw}(Y)$, where

$$b = \|g\| (1 + 2/(c - w) + (c + w)/(c - w)) \tag{8-7}$$

Multiplying the expressions in inequality (8-6) by e^{-kt} and taking the supremum over all $t \in I$, we get the inequality

$$|u_1 - u_2|_k \leq b|y_1 - y_2|_k \tag{8-8}$$

for all $k \geq 0$ and $y_1, y_2 \in L_{xw}(Y)$. The last inequality proves that

$$\|U\|_k \leq b \tag{8-9}$$

for all $k \geq 0$; i.e., the operator U is of the exponential type from every set $L_{xw}(Y)$ into the space $C(Z)$ of continuous functions.

The above proves that the conditions of the theorem in section 7.0 are satisfied, and thus for every initial condition

$$y(t) = x(t) \quad \text{for} \quad t \leq 0 \tag{8-3}$$

there exists a unique solution to the automatic control problem given by formulas (8-1) and (8-2). This concludes the proof of the theorem.

9.0 CONCLUSION

The main result of this paper is the proof of the existence and uniqueness of the solution to the automatic control problem with nonlinear state equation of the form

$$y' = f(t,y,u) \tag{9-1}$$

and nonlinear operator controls

$$u = U(y) \tag{9-2}$$

acting onto the state function y which satisfies the initial condition

$$y(t) = x(t) \tag{9-3}$$

for $t \leq 0$.

The required regularity conditions on the function f and the operator U are the following: The function is of Lipschitzian type in the variables y and u and is bounded and continuous on its domain. The operator U is Lipschitzian from every set $L_{xw}(Y)$ into the space of continuous functions for every initial function x with Lipschitz constant $w = \|f\|$. Moreover, the induced seminorm $\|U\|_k$ as a function of k should be bounded for sufficiently large k. This class of nonlinear operators is said to be of the exponential type. Generalization of these results will appear in reference 8.

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16. Abstract

The main result of this paper is the proof of the existence and uniqueness of the solution to the automatic control problem with a nonlinear state equation of the form $y' = f(t,y,u)$ and nonlinear operator controls $u = U(y)$ acting onto the state function y which satisfies the initial condition $y(t) = x(t)$ for $t \leq 0$.

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